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集中载荷作用下四直边上任意点支承矩形板的弯曲

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摘　要:基于薄板弯曲问题的广义简支边界条件,通过将集中载荷作用下四直边上任意点支承矩形板的弯曲问题分解为 6 个基本的薄板弯曲问题,应用叠加法首次得到了该问题的解析解.对典型的薄板弯曲问题进行了计算分析,取得了较好的效果.该结果可以作为基本解用于求解任意荷载作用下四直边上任意点支承矩形板的弯曲问题.

关键词:点支承矩形板;弯曲;广义简支边界
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0　引言

四直边上任意点支承的矩形板的弯曲问题在机械、土建等工程问题中有着广泛的应用,例如高台建筑等,研究这一问题具有重要的实际意义.文献[1]曾采用叠加法对均布载荷作用下一边固定其对边中点被支承,而另两边自由的矩形板弯曲问题进行了研究.文献[2]应用瑞雷 - 李兹法,通过假设容许位移,求解了在均布载荷作用下四直边中点支承方形板的弯曲问题.由于猜测容许位移的方法是没有普遍意义的,并且求出的是近似解,还需要研究其近似程度.本文基于薄板弯曲的广义简支边界条件[3],将集中载荷作用下四直边上任意点支承矩形板的弯曲问题分解为 6 个基本的薄板弯曲问题,应用叠加法给出了问题的解析解.本文的求解方法能够较容易地推广到求解其它类似问题.

1　理论分析

对于任意一个矩形薄板,假定其边长为 a 和 b,在任意一点(x_0, y_0)处作用有集中载荷 P,在其四条边上的任意一点处作用支承约束,其位置如图 1 所示.

根据弹性理论,薄板的控制方程为

$$D\nabla^4 W = P\delta(x - x_0, y - y_0)\ . \qquad (1)$$

式中:$D = \dfrac{Eh^3}{12(1-\nu^2)}$;$E$ 为弹性模量;ν 为泊松比;h 为板的厚度;W 为板的挠度.

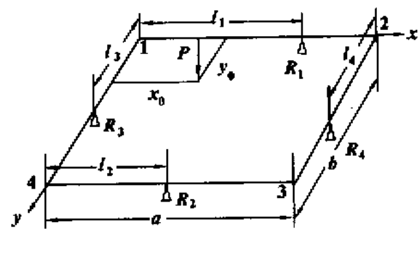

图 1　集中载荷作用下四直边上任意点支承的矩形板

Fig.1　A rectangular plate supported at any point on the straight edges under concentrated load

根据文献[2]的有关结果,该问题的边界条件可以写为以下形式:

$$-D\left[\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2}\right]_{x=0} = 2\frac{R_3}{b}\sum_{n=1}^{\infty}\sin\beta_n l_3 \sin\beta_n y\ ; \qquad (2)$$

$$-D\left[\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2}\right]_{x=a} = -2\frac{R_4}{b}\sum_{n=1}^{\infty}\sin\beta_n l_4 \sin\beta_n y\ ; \qquad (3)$$

$$-D\left[\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial x^2 \partial y}\right]_{y=0} =$$

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$$2\frac{R_1}{a}\sum_{m=1}^{\infty}\sin\alpha_m l_1\sin\alpha_m x\ ; \tag{4}$$

$$-D\left[\frac{\partial^3 W}{\partial y^3}+(2-\nu)\frac{\partial^3 W}{\partial x^2\partial y}\right]_{y=b}=$$

$$-2\frac{R_2}{a}\sum_{m=1}^{\infty}\sin\alpha_m l_2\sin\alpha_m x\ ; \tag{5}$$

$$W_{x=0,y=l_3}=0\ ; \tag{6}$$

$$W_{x=a,y=l_4}=0\ ; \tag{7}$$

$$W_{x=l_1,y=0}=0\ ; \tag{8}$$

$$W_{x=l_2,y=0}=0\ . \tag{9}$$

式中：R_1 和 R_2，R_3 和 R_4 分别为支承点的支反力；$\alpha_m=\frac{m\pi}{a}$；$\beta_n=\frac{n\pi}{b}$. 此外，由于薄板的 4 个角点均自由，其集中力为零. 因此，在角点上还应满足以下条件：

$$-2D(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)_{x=0,y=0}=0\ ; \tag{10}$$

$$-2D(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)_{x=a,y=0}=0\ ; \tag{11}$$

$$-2D(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)_{x=a,y=b}=0\ ; \tag{12}$$

$$-2D(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)_{x=0,y=b}=0\ . \tag{13}$$

一般来说，边值问题(1) ~ (13)的求解是十分困难的. 我们基于薄板弯曲问题的广义简支边界条件，将该问题分解为以下 6 个基本问题分别进行求解. 在此基础上，应用叠加法可以得到该问题的解答.

(1) 假定矩形板的四边均为简支，在点(x_0，y_0)处作用集中载荷 P，则板的弯曲面为

$$W_P=\frac{P}{Db}\sum_{n=1}^{\infty}[(1+\beta_n a\operatorname{cth}\beta_n a)-\beta_n x\operatorname{cth}\beta_n x-\beta_n(a-x_0)\operatorname{cth}\beta_n(a-x_0)]\frac{1}{\beta_n^3\operatorname{sh}\beta_n a}\cdot\operatorname{sh}\beta_n x\cdot\operatorname{sh}\beta_n(a-x_0)\sin\beta_n y_0\sin\beta_n y\quad(0\le x\le x_0)\ ; \tag{14}$$

$$W_P=\frac{P}{Db}\sum_{n=1}^{\infty}[(1+\beta_n a\operatorname{cth}\beta_n a)-\beta_n(a-x)\operatorname{cth}\beta_n(a-x)-\beta_n x_0\operatorname{cth}\beta_n x_0]\frac{1}{\beta_n^3\operatorname{sh}\beta_n a}\cdot\operatorname{sh}\beta_n(a-x)\operatorname{sh}\beta_n x_0\sin\beta_n y_0\sin\beta_n y\quad(x_0<x\le a)\ ; \tag{15}$$

或者为另一组等价的表达式

$$W_P=\frac{P}{Da}\sum_{m=1}^{\infty}[(1+\alpha_m b\operatorname{cth}\alpha_m b)-\alpha_m y\operatorname{cth}\alpha_m y-\alpha_m(b-y_0)\operatorname{cth}\alpha_m(b-y_0)]\frac{1}{\alpha_m^3\operatorname{sh}\alpha_m b}\cdot\operatorname{sh}\alpha_m b\cdot\operatorname{sh}\alpha_m(b-y_0)\sin\alpha_m x_0\cdot\sin\alpha_m x\quad(0\le y\le y_0)\ ; \tag{16}$$

$$W_P=\frac{P}{Da}\sum_{m=1}^{\infty}[(1+\alpha_m b\operatorname{cth}\alpha_m b)-\alpha_m(b-y)\operatorname{cth}\alpha_m(b-y)-\alpha_m y_0\operatorname{cth}\alpha_m y_0]\frac{1}{\alpha_m^3\operatorname{sh}\alpha_m b}\cdot\operatorname{sh}\alpha_m(b-y)\operatorname{sh}\alpha_m y_0\sin\alpha_m x_0\sin\alpha_m x\quad(y_0\le y\le b)\ . \tag{17}$$

(2) 假定板边 $x=0$ 为广义简支，其它三边均为简支，并设沿 $x=0$ 边上的挠度为

$$W_{x=0}=\sum_{n=1}^{\infty}\xi_n\sin\beta_n y\ . \tag{18}$$

式中：ξ_n 为待定系数，则板的弯曲面为

$$W_1=\frac{1}{2}\sum_{n=1}^{\infty}\{2+(1-\nu)[\beta_n a\operatorname{cth}\beta_n a-\beta_n(a-x)\operatorname{cth}\beta_n(a-x)]\}\frac{\xi_n}{\operatorname{sh}\beta_n a}\operatorname{sh}\beta_n(a-x)\sin\beta_n y\ . \tag{19}$$

(3) 假定板边 $x=a$ 为广义简支，其它三边均为简支，并设沿 $x=a$ 边上的挠度为

$$W_{x=a}=\sum_{n=1}^{\infty}\zeta_n\sin\beta_n y\ . \tag{20}$$

式中：ζ_n 为待定系数，则板的弯曲面为

$$W_2=\frac{1}{2}\sum_{n=1}^{\infty}\{2+(1-\nu)[\beta_n a\operatorname{cth}\beta_n a-\beta_n x\operatorname{cth}\beta_n x]\}\frac{\zeta_n}{\operatorname{sh}\beta_n a}\operatorname{sh}\beta_n x\sin\beta_n y\ . \tag{21}$$

(4) 假定板边 $y=0$ 为广义简支，其它三边均为简支，并设沿 $y=0$ 边上的挠度为

$$W_{y=0}=\sum_{m=1}^{\infty}c_m\sin\alpha_m x\ . \tag{22}$$

式中：c_m 为待定系数，则板的弯曲面为

$$W_3=\frac{1}{2}\sum_{m=1}^{\infty}\{2+(1-\nu)[\alpha_m b\operatorname{cth}\alpha_m b-\alpha_m(b-y)\operatorname{cth}\alpha_m(b-y)]\}\frac{c_m}{\operatorname{sh}\alpha_m b}\operatorname{sh}\alpha_m(b-y)\sin\alpha_m x\ . \tag{23}$$

(5) 假定板边 $y=b$ 为广义简支，其它三边均为简支，并设沿 $y=b$ 边上的挠度为

$$W_{y=b}=\sum_{m=1}^{\infty}d_m\sin\alpha_m x\ . \tag{24}$$

式中：d_m 为待定系数，则板的弯曲面为

$$W_4=\frac{1}{2}\sum_{m=1}^{\infty}\{2+(1-\nu)[\alpha_m b\operatorname{cth}\alpha_m b-$$

$$\alpha_m y \operatorname{cth}\alpha_m y]\}\frac{d_m}{\operatorname{sh}\alpha_m b}\operatorname{sh}\alpha_m y\sin\alpha_m x\ . \tag{25}$$

(6) 由于板的 4 个角点都是自由角点,所以各边都有刚体位移,其对应的板的弯曲面为

$$W_5 = \frac{a-x}{a}\frac{b-y}{b}k_1 + \frac{x}{a}\frac{b-y}{b}k_2 + \frac{x}{a}\frac{y}{b}k_3 + \frac{a-x}{a}\frac{y}{b}k_4. \tag{26}$$

式中:k_1, k_2, k_3 和 k_4 为待定常数.

集中载荷作用下四直边上任意点支承矩形板弯曲问题的最终解答为上述 6 种情况的叠加,故

$$W = W_P + W_1 + W_2 + W_3 + W_4 + W_5. \tag{27}$$

将式(27)依次代入边界条件(2)~(13),可以得到如下方程组

$$\frac{P}{b}\{2 + (1-\nu)[\beta_n a\operatorname{cth}\beta_n a - \beta_n(a-x_0)\operatorname{cth}\beta_n(a-x_0)]\}\frac{1}{\operatorname{sh}\beta_n a}\operatorname{sh}\beta(a-x_0)\sin\beta_n y_0 - \frac{D}{2}[2(1-\nu^2)\operatorname{ch}\beta_n a + (1-\nu)^2(\operatorname{ch}\beta_n a + \frac{\beta_n a}{\operatorname{sh}\beta_n a})]\frac{\beta_n^3}{\operatorname{sh}\beta_n a}(\xi_n) + \frac{D}{2}[2(1-\nu^2) + (1-\nu^2)(1+\beta_n a\operatorname{cth}\beta_n a)]\cdot\frac{\beta_n^3}{\operatorname{sh}\beta_n a}(\zeta_n) + \frac{2D}{b}(1-\nu)^2\sum_{m=1}^{\infty}\frac{\alpha_m^3\beta_n^3}{(\alpha_m^2+\beta_n^2)^2}(c_m) + \frac{2D}{b}(1-\nu)^2\sum_{m=1}^{\infty}\frac{\alpha_m^3\beta_n^3}{(\alpha_m^2+\beta_n^2)^2}(-1)^{n+1}(d_m) = 2\frac{R_3}{b}\sin\beta_n l_3 \quad (n = 1,2,\cdots)\ ; \tag{28}$$

$$-\frac{P}{b}[2 + (1-\nu)(\beta_n a\operatorname{cth}\beta_n a - \beta_n x_0\operatorname{cth}\beta_n x_0)]\cdot\frac{1}{\operatorname{sh}\beta_n a}\operatorname{sh}\beta_n x_0\sin\beta_n y_0 - \frac{D}{2}[2(1-\nu^2) + (1-\nu)^2(1+\beta_n a\operatorname{cth}\beta_n a)]\frac{\beta_n^3}{\operatorname{sh}\beta_n a}(\xi_n) + \frac{D}{2}[2(1-\nu^2)\operatorname{ch}\beta_n a + (1-\nu)^2(\operatorname{ch}\beta_n a + \frac{\beta_n a}{\operatorname{sh}\beta_n a})]\frac{\beta_n^3}{\operatorname{sh}\beta_n a}(\zeta_n) + \frac{2D}{b}(1-\nu)^2\sum_{m=1}^{\infty}\frac{\alpha_m^3\beta_n^3}{(\alpha_m^2+\beta_n^2)^2}(-1)^m(c_m) + \frac{2D}{b}(1-\nu)^2\cdot\sum_{m=1}^{\infty}\frac{\alpha_m^3\beta_n^3}{(\alpha_m^2+\beta_n^2)^2}(-1)^{m+n+1}(d_m) = -2\frac{R_4}{b}\sin\beta_n l_4 \quad (n = 1,2,\cdots)\ ; \tag{29}$$

$$\frac{P}{a}\{2 + (1-\nu)[\alpha_m b\operatorname{cth}\alpha_m b - \alpha_m(b-y_0)\operatorname{cth}\alpha_m(b-y_0)]\}\frac{1}{\operatorname{sh}\alpha_m b}\operatorname{sh}\alpha_m(b-y_0)\sin\alpha_m x_0 - \frac{2D}{a}(1-\nu)^2\cdot\sum_{n=1}^{\infty}\frac{\alpha_m^3\beta_n^3}{(\alpha_m^2+\beta_n^2)^2}(\xi_n) + \frac{2D}{a}(1-\nu)^2\sum_{n=1}^{\infty}\frac{\alpha_m^3\beta_n^3}{(\alpha_m^2+\beta_n^2)^2}\cdot(-1)^{m+1}(\zeta_n) - \frac{D}{2}[2(1-\nu^2)\operatorname{ch}\alpha_m b + (1-\nu)^2(\operatorname{ch}\alpha_m b + \frac{\alpha_m b}{\operatorname{sh}\alpha_m b})]\frac{\alpha_m^3}{\operatorname{sh}\alpha_m b}(c_m) - \frac{D}{2}[2(1-\nu^2) + (1-\nu)^2(1-\alpha_m b\operatorname{cth}\alpha_m b)]\frac{\alpha_m^3}{\operatorname{sh}\alpha_m b}(d_m) = 2\frac{R_1}{a}\sin\alpha_m l_1 \quad (m = 1,2,\cdots)\ ; \tag{30}$$

$$-\frac{P}{a}[2 + (1-\nu)(\alpha_m b\operatorname{cth}\alpha_m b - \alpha_m y_0\operatorname{cth}\alpha_m y_0)]\cdot\frac{1}{\operatorname{sh}\alpha_m b}\operatorname{sh}\alpha_m y_0\sin\alpha_m x_0 - \frac{2D}{a}(1-\nu)^2\sum_{n=1}^{\infty}\frac{\alpha_m^3\beta_n^3}{(\alpha_m^2+\beta_n^2)^2}\cdot(-1)^n(\xi_n) + \frac{2D}{a}(1-\nu)^2\sum_{n=1}^{\infty}\frac{\alpha_m^3\beta_n^3}{(\alpha_m^2+\beta_n^2)^2}(-1)^{m+n+1}(\zeta_n) - \frac{D}{2}[2(1-\nu^2) + (1-\nu)^2(1-\alpha_m b\operatorname{cth}\alpha_m b)]\frac{\alpha_m^3}{\operatorname{sh}\alpha_m b}(c_m) - \frac{D}{2}[2(1-\nu^2)\operatorname{ch}\alpha_m b + (1-\nu)^2(\operatorname{ch}\alpha_m b + \frac{\alpha_m b}{\operatorname{sh}\alpha_m b})]\frac{\alpha_m^3}{\operatorname{sh}\alpha_m b}(d_m) = 2\frac{R_2}{a}\sin\alpha_m l_2 \quad (m = 1,2,\cdots)\ ; \tag{31}$$

$$\sum_{n=1}^{\infty}(\xi_n)\sin\beta_n l_3 + \frac{b-l_3}{b}k_1 + \frac{l_3}{b}k_4 = 0\ ; \tag{32}$$

$$\sum_{n=1}^{\infty}(\zeta_n)\sin\beta_n l_4 + \frac{b-l_4}{b}k_2 + \frac{l_4}{b}k_3 = 0\ ; \tag{33}$$

$$\sum_{m=1}^{\infty}(c_m)\sin\alpha_m l_1 + \frac{a-l_1}{a}k_1 + \frac{l_1}{a}k_2 = 0\ ; \tag{34}$$

$$\sum_{m=1}^{\infty}(d_m)\sin\alpha_m l_2 + \frac{a-l_2}{a}k_4 + \frac{l_1}{a}k_3 = 0\ ; \tag{35}$$

$$-2(1-\nu)\frac{P}{b}\sum_{n=1}^{\infty}[a\operatorname{cth}\beta_n a - (a-x_0)\operatorname{cth}\beta_n(a-x_0)]\frac{1}{\operatorname{sh}\beta_n a}\operatorname{sh}\beta_n(a-x_0)\sin\beta_n y_0 + D(1-\nu)\sum_{n=1}^{\infty}[(\operatorname{ch}\beta_n a + \frac{\beta_n a}{\operatorname{sh}\beta_n a}) + \nu(\operatorname{ch}\beta_n a - \frac{\beta_n a}{\operatorname{sh}\beta_n a})]\cdot\frac{\beta_n^2}{\operatorname{sh}\beta_n a}(\xi_n) - D(1-\nu)\sum_{n=1}^{\infty}[(1+\beta_n a\operatorname{cth}\beta_n a) + \nu(1-\beta_n a\operatorname{cth}\beta_n a)]\frac{\beta_n^2}{\operatorname{sh}\beta_n a}(\zeta_n) + D(1-\nu)\sum_{m=1}^{\infty}[(\operatorname{ch}\alpha_m b + \frac{\alpha_m b}{\operatorname{sh}\alpha_m b}) + \nu(\operatorname{ch}\alpha_m b - \frac{\alpha_m b}{\operatorname{sh}\alpha_m b})]\frac{\alpha_n^2}{\operatorname{sh}\alpha_m b}(c_m) - D(1-\nu)\sum_{m=1}^{\infty}[(1+\alpha_m b\operatorname{cth}\alpha_m b) + \nu(1-\alpha_m b\operatorname{cth}\alpha_m b)]\cdot\frac{\alpha_m^2}{\operatorname{sh}\alpha_m b}(d_m) - 2D(1-\nu)\frac{1}{ab}(k_1-k_2+k_3-k_4) = 0\ ; \tag{36}$$

$$2(1-\nu)\frac{P}{b}\sum_{n=1}^{\infty}[a\operatorname{cth}\beta_n a - x_0\operatorname{cth}\beta_n x_0]\frac{1}{\operatorname{sh}\beta_n a}\operatorname{sh}\beta_n\cdot x_0\sin\beta_n y_0 + D(1-\nu)\sum_{n=1}^{\infty}[(1+\beta_n a\operatorname{cth}\beta_n a) + \nu(1-$$

$$\beta_n a\operatorname{cth}\beta_n a)]\cdot\frac{\beta_n^2}{\operatorname{sh}\beta_n a}(\xi_n)-D(1-\nu)\sum_{n=1}^{\infty}[(\operatorname{ch}\beta_n a-\frac{\beta_n a}{\operatorname{sh}\beta_n a})+\nu(\operatorname{ch}\beta_n a-\frac{\beta_n a}{\operatorname{sh}\beta_n a})]\frac{\beta_n^2}{\operatorname{sh}\beta_n a}(\zeta_n)+D(1-\nu)\sum_{m=1}^{\infty}[(\operatorname{ch}\alpha_m b+\frac{\alpha_m b}{\operatorname{sh}\alpha_m b})+\nu(\operatorname{ch}\alpha_m b-\frac{\alpha_m b}{\operatorname{sh}\alpha_m b})]\cdot\frac{\alpha_m^2}{\operatorname{sh}\alpha_m b}(-1)^m(c_m)-D(1-\nu)\sum_{m=1}^{\infty}[(1+\alpha_m b\operatorname{cth}\alpha_m b)+\nu(1-\alpha_m b\operatorname{cth}\alpha_m b]\frac{\alpha_m^2}{\operatorname{sh}\alpha_m b}(-1)^m(d_m)-2D(1-\nu)\frac{1}{ab}(k_1-k_2+k_3-k_4)=0\ ; \qquad (37)$$

$$2(1-\nu)\frac{P}{b}\sum_{n=1}^{\infty}[a\operatorname{cth}\beta_n a-x_0\operatorname{cth}\beta_n x_0]\frac{(-1)^n}{\operatorname{sh}\beta_n a}\cdot\operatorname{sh}\beta_n x_0\sin\beta_n y_0+D(1-\nu)\sum_{n=1}^{\infty}[(1+\beta_n a\operatorname{cth}\beta_n a)+\nu(1-\beta_n a\operatorname{cth}\beta_n a)]\frac{\beta_n^2}{\operatorname{sh}\beta_n a}(-1)^n(\xi_n)-D(1-\nu)\sum_{n=1}^{\infty}[(\operatorname{ch}\beta_n a+\frac{\beta_n a}{\operatorname{sh}\beta_n a})+\nu(\operatorname{ch}\beta_n a-\frac{\beta_n a}{\operatorname{sh}\beta_n a})]\cdot\frac{\beta_n^2}{\operatorname{sh}\beta_n a}(-1)^n(\zeta_n)+D(1-\nu)\sum_{m=1}^{\infty}[(1+\alpha_m b\operatorname{cth}\alpha_m b)+\nu(1-\alpha_m b\operatorname{cth}\alpha_m b)]\frac{\alpha_m^2}{\operatorname{sh}\alpha_m b}(-1)^m(c_m)-D(1-\nu)\sum_{m=1}^{\infty}[(\operatorname{ch}\alpha_m b+\frac{\alpha_m b}{\operatorname{sh}\alpha_m b})]+\nu(\operatorname{ch}\alpha_m b-\frac{\alpha_m b}{\operatorname{sh}\alpha_m b})]\cdot\frac{\alpha_m^2}{\operatorname{sh}\alpha_m b}(-1)^m(d_m)-2D(1-\nu)\frac{1}{ab}(k_1-k_2+k_3-k_4)=0\ ; \qquad (38)$$

$$-2(1-\nu)\frac{P}{b}\sum_{n=1}^{\infty}[a\operatorname{cth}\beta_n a-(a-x_0)\operatorname{cth}\beta_n(a-x_0)]\frac{(-1)^n}{\operatorname{sh}\beta_n a}\operatorname{sh}\beta_n(a-x_0)\sin\beta_n y_0+D(1-\nu)\sum_{n=1}^{\infty}[(\operatorname{ch}\beta_n a+\frac{\beta_n a}{\operatorname{sh}\beta_n a})+\nu(\operatorname{ch}\beta_n a-\frac{\beta_n a}{\operatorname{sh}\beta_n a})]\cdot\frac{\beta_n^2}{\operatorname{sh}\beta_n a}(-1)^n(\xi_n)-D(1-\nu)\sum_{n=1}^{\infty}[(1+\beta_n a\operatorname{cth}\beta_n a)+\nu(1-\beta_n a\operatorname{cth}\beta_n a)]\frac{\beta_n^2}{\operatorname{sh}\beta_n a}(-1)^n(\zeta_n)+D(1-\nu)\sum_{m=1}^{\infty}[(1+\alpha_m b\operatorname{cth}\alpha_m b)+\nu(1-\alpha_m b\operatorname{cth}\alpha_m b)]\cdot\frac{\alpha_m^2}{\operatorname{sh}\alpha_m b}(c_m)-D(1-\nu)\sum_{m=1}^{\infty}[(\operatorname{ch}\alpha_m b+\frac{\alpha_m b}{\operatorname{sh}\alpha_m b})+\nu(\operatorname{ch}\alpha_m b-\frac{\alpha_m b}{\operatorname{sh}\alpha_m b})]\frac{\alpha_m^2}{\operatorname{sh}\alpha_m b}(d_m)-2D(1-\nu)\frac{1}{ab}(k_1-k_2+k_3-k_4)=0\ . \qquad (39)$$

求解线性方程组(28)～(39),可以得到诸常数 $\xi_n,\zeta_n,c_m,d_m,k_1,k_2,k_3,k_4,R_1,R_2,R_3,R_4$,将结果代入式(27),可以进一步可求得板的挠度、剪力、弯矩等值.

2 数值算例

在上述求解过程中,因为四条边都是自由边,所以其收敛速度较慢.计算结果表明,对于不同的矩形薄板,上述方程组中,m 和 n 的取值在 40 至 80 之间即可得到具有较高精度的计算结果.为此,在求解过程中,本文分别取 $m=n=80$.另外,取 $\nu=0.3$,且令 $c=\min\{a,b\}$.

作为数值算例,本文重点对在矩形板的中心位置作用集中力,且其四边中点支承的矩形板弯曲问题进行了研究,分别计算了 $b/a=1.0, 2.0, 0.5$ 三种情况.表 1 分别给出了采用本文和有限元方法(Super SAP)计算正方形薄板挠度系数 $W(Pc^2/D)$ 的结果.由此可以看出,两种结果符合较好.表 2 给出了三种矩形薄板沿 $x=0$ 边的挠度系数 $W(Pc^2/D)$,表 3 给出了其沿直线 $x=a/2$ 的挠度系数 $W(Pc^2/D)$、弯矩系数 $M_x(P)$ 和 $M_y(P)$.由于在集中载荷作用点的附近区域,板的内力是奇异的,因此,在表 2 中集中载荷作用点($x=a/2, y=b/2$)的附近区域,M_x 和 M_y 的值较其它点的值大很多.

表 1 正方形薄板的挠度系数($\times 10^{-2}$)

Table 1 Square plate deflection factors($\times 10^{-2}$)

y/b	$x=0$		$x=a/2$	
	本文	有限元	本文	有限元
0.0	-0.318	-0.304	0.000	0.000
0.1	-0.166	-0.155	0.386	0.360
0.2	-0.042	-0.040	0.694	0.688
0.3	0.023	0.022	1.016	0.980
0.4	0.028	0.026	1.241	1.216
0.5	0.000	0.000	1.352	1.346

表 2 沿 $x=0$ 边的挠度系数($\times 10^{-2}$)

Table 2 Deflection factors along the side $x=0$($\times 10^{-2}$)

y/b	$b/a=1.0$	$b/a=2.0$	$b/a=0.5$
0.0	-0.318	-0.163	-0.163
0.1	-0.166	0.143	-0.092
0.2	-0.042	0.348	-0.038
0.3	0.023	0.396	-0.012
0.3	0.023	0.396	-0.012
0.4	0.028	0.237	0.013
0.5	0.000	0.000	0.000

表 3 沿直线 $x = a/2$ 的挠度系数和弯矩系数

Table 3 Deflection factors and moment factors along the side $x = a/2$

y/a	$b/a=1.0$			$b/a=2.0$			$b/a=0.5$		
	W	M_x	M_y	W	M_x	M_y	W	M_x	M_y
0.0	0.000	-0.206	0.00	0.000	-0.050	0.00	0.00	-0.378	0.00
0.1	0.004	0.030	0.033	0.004	0.028	0.011	0.006	-0.047	0.051
0.2	0.007	0.085	0.057	0.009	0.056	0.020	0.011	0.034	0.082
0.3	0.010	0.137	0.094	0.013	0.098	0.034	0.015	0.099	0.125
0.4	0.012	0.214	0.161	0.017	0.176	0.073	0.018	0.181	0.197
0.5	0.013	0.632	0.632	0.019	0.597	0.600	0.019	0.600	0.597

3 结束语

对于在集中载荷作用下四直边上任意点支承矩形板的弯曲问题,本文基于薄板弯曲的广义简支边界条件,应用叠加法给出了问题的精确解的表达式.由于该方法能够得到精度较高的计算结果,不仅可以用于考核其它计算该类问题数值方法的收敛问题,而且可用于指导此类结构的分析与设计,具有一定的理论意义和实用价值.本文所采用的求解方法思路清晰,能够较容易地推广用于求解其它类似的薄板弯曲问题.

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Bending of Rectangular Plates Supported at Any Point on the Straight Edges under a Concentrated Load

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Abstract: In this paper, the general supported boundary condition is applied to solve the bending of rectangular plates supported at any point on the four straight edges under a concentrated load. Firstly, the problem is reduced to solving six fundamental rectangular plate bending problems, and then, an analytical solution of the problem is derived by using superposition method. By authors known, it is the first time to give the analytical solution of the problem. Finally, some typical problems are calculated, and the results are prove to be satisfacory. The results may be used as a fundamental solution to analyse the bending of rectangular plates supported at any point on the straight edges under arbitrary loads.

Key words: rectangular plate supported at points; bending; general supported boundary