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# Stress Singularity and Stress Field for a Crack Terminating at Tri – phase Interface

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**Abstract**: In this paper, we investigate the dependence of the order of stress singularity at the tip of a crack, which terminates at the vertex of a tri-material wedge, on its geometry and the elastic constants. The stress field formulation near the crack tip in a system of polar coordinates which origin at the singular point is derived in terms of complex stress intensity factor $K$ by using complex potentials. Several exampels prove that the solution of this paper is correct and some useful conclusions are obtained.

**Key words**: stress singularity; interfacial crack; complex variable method

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## Introduction

In recent years, the use of composite material has been given considerable attention in structural applications[1~3]. The appearance of flaws or crackson the interface between the two materials could reduce the strength of the structure significantly. Several authors have investigated the nature of the stress singularity at the tip of a crack between dissimilar media. The order of this singularity (which is $r^{-1/2}$ for isotropic materials) is dependent on both the crack geometry and the material constants.

In present study, we first deal with the stress singularity at the tip of a crack on a interface of a tri-material wedge, and derive the stress field formulations in complex variable method and elasticity theory. Finally numerical results are discussed.

## 1 Analytical Development

Consider a plane region sketched in Fig.1, consisting of three wedges with different material elastic constants. Assume that a crack lays on a interface of two wedges and terminates at the apex of the tri-material wedge. A polar coordinate system is established as Fig.1 shows, distinguishing three regions with a subscript $\alpha$, where $\alpha = 1, 2$, and 3 respectively coorecorresponds to $0 \leqslant \theta \leqslant \pi$, $-\pi \leqslant \theta \leqslant \beta$ and $\beta \leqslant \theta \leqslant 0$. In the absence of body force and in terms of complex potentials $\Omega_\alpha(z)$ and $\omega_\alpha(z)$, the displacement $u_\alpha$ and stresses $\sigma_r^\alpha$, $\sigma_\theta^\alpha$ are given by

$$
\begin{aligned}
2\mu_\alpha \mu_\alpha &= 2\mu_\alpha(u_r^\alpha + iu_\theta^\alpha) = e^{i\theta}\{k_\alpha \Omega_\alpha(z) - 2\bar{\Omega}_\alpha(\bar{z}) - \bar{\omega}_\alpha(\bar{z})\}; \\
\sigma_r^\alpha &= \sigma_{rr}^\alpha + \sigma_{r\theta}^\alpha = \Omega'_\alpha(z) + \bar{\Omega}'_\alpha(\bar{z}) - \bar{z}\bar{\Omega}''_\alpha(\bar{z}) - \frac{\bar{z}}{z}\bar{\omega}'_\alpha(\bar{z}); \\
\sigma_\theta^\alpha &= \sigma_{\theta\theta}^\alpha - i\sigma_{r\theta}^\alpha = \Omega'_\alpha(z) + \bar{\Omega}'_\alpha(\bar{z}) + \bar{z}\bar{\Omega}''_\alpha(\bar{z}) + \frac{\bar{z}}{z}\bar{\omega}'_\alpha(\bar{z}),
\end{aligned}
\tag{1}
$$

where $z = re^{i\theta}$, $\mu_\alpha$ is rigidity modulus, and $k_\alpha = 3 - 4\nu_\alpha$ for plane strain, $(3 - \nu_\alpha)/(1 + \nu_\alpha)$ for plane stress, $\nu_\alpha$ being posson's ratio.

With the assumption of traction-free on the crack surfaces at $\theta = \pm\pi$, the following boundray conditions must be satisfied:

$$
\begin{aligned}
&\text{at } \theta = \pi, \sigma_\theta^1 = 0; \\
&\text{at } \theta = -\pi, \sigma_\theta^2 = 0; \\
&\text{at } \theta = 0, \sigma_\theta^1 = \sigma_\theta^3, u^1 = u^3; \\
&\text{at } \theta = \beta, \sigma_\theta^2 = \sigma_\theta^3, u^2 = u^3,
\end{aligned}
\tag{2}
$$

Following the procedure used by England[4] and C.C

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Hong and M. Stern[5], we look for potentials of the form:

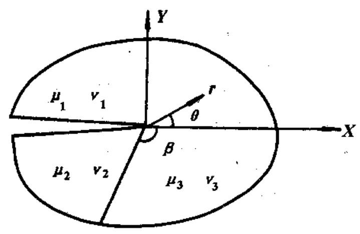


**Fig.1 Coordinate system with respect to the crack tip in a tri-material wedge**

$$\Omega_\alpha(z) = A_\alpha z^\lambda, \quad \omega_\alpha(z) = B_\alpha z^\lambda. \tag{3}$$

where $A_\alpha$ and $B_\alpha$ are complex constants.

Demanding the existence of nontrivial solutions satisfying the boundary conditions[5] leads to the characteristic equation:

$$f(\lambda) = | d_{ij} | = 0 \quad (i, j = 1,6), \tag{4}$$

where

$$\begin{aligned}
d_{11} &= (\mu_3/\mu_1)(k_1 + e^{2i\lambda\pi}) + (1 - e^{2i\lambda\pi}),\\
d_{12} &= d_{13} = d_{14} = d_{16} = d_{21} = d_{23} = \\
&\quad d_{24} = d_{25} = 0,\\
d_{15} &= d_{26} = -(k_3 + 1),\\
d_{22} &= (\mu_3/\mu_1)(k_1 + e^{-2i\lambda\pi}) + (1 - e^{-2i\lambda\pi}),\\
d_{31} &= d_{51} = d_{61} = 1 - e^{2i\lambda\pi},\\
d_{32} &= d_{41} = d_{52} = d_{62} = 0,\\
d_{33} &= (\mu_3/\mu_2)(k_2 e^{2i\lambda\beta} + e^{-2i\lambda\pi}),\\
d_{34} &= (\lambda\mu_3/\mu_2)(1 - e^{2i\beta}),\\
d_{35} &= -(k_3 e^{2i\lambda\beta} + 1), d_{36} = \lambda(e^{2i\beta} - 1),\\
d_{42} &= 1 - e^{-2i\lambda\pi}, d_{43} = (\lambda\mu_3/\mu_2)(1 - e^{2i\beta}),\\
d_{44} &= (\mu_3/\mu_2)(k_2 e^{-2i\lambda\beta} + e^{2i\lambda\pi}),\\
d_{45} &= d_{65} = \lambda(e^{-2i\beta} - 1),\\
d_{46} &= -(k_3 e^{-2i\lambda\beta} + 1), d_{53} = -e^{2i\lambda\beta} + e^{-2i\lambda\pi},\\
d_{54} &= \lambda(1 - e^{2i\beta}), d_{55} = e^{2i\lambda\beta} - 1,\\
d_{56} &= \lambda(e^{2i\beta} - 1), d_{63} = \lambda(1 - e^{-2i\beta}),\\
d_{64} &= e^{2i\lambda\pi} - e^{-2i\lambda\beta}, d_{66} = e^{-2i\lambda\beta} - 1.
\end{aligned} \tag{5}$$

The eigenvalue $\lambda$ can be found by the solution of the equation (4).

We notice that these eigenvalues occur in possitive and negative pairs. To represent a physically meaningful singular state, Re $\lambda$ must lie between zero and one. therefore the complex potential of the singular state are then of the form

$$\Omega_\alpha = A_\alpha z^\lambda + a_\alpha z^{\bar\lambda},$$

$$\omega_\alpha = B_\alpha z^\lambda + b_\alpha z^{\bar\lambda}, \tag{6}$$

so the equation (1) becomes the following form:

$$\begin{aligned}
2\mu_\alpha u_\alpha &= r^\lambda(k_\alpha A_\alpha e^{i(\lambda-1)\theta} - \bar a_\alpha \lambda e^{-i(\lambda-1)\theta} - \\
&\quad \bar b_\alpha e^{-i(\lambda+1)\theta}) + r^{\bar\lambda}(k_\alpha a_\alpha e^{i(\bar\lambda-1)\theta} - \\
&\quad \bar A_\alpha \lambda e^{-i(\bar\lambda-1)\theta} - \bar B_\alpha e^{-i(\bar\lambda+1)\theta}),\\
\sigma_r^\alpha &= \lambda r^{\lambda-1}[A_\alpha e^{i(\lambda-1)\theta} - \bar a_\alpha(\lambda - 2)e^{-i(\lambda-1)\theta} - \\
&\quad \bar b_\alpha e^{-i(\lambda+1)\theta}] + \bar\lambda r^{\bar\lambda-1}[a_\alpha e^{i(\bar\lambda-1)\theta} - \\
&\quad \bar A_\alpha(\bar\lambda - 2)e^{-i(\bar\lambda-1)\theta} - \bar B_\alpha e^{-i(\bar\lambda+1)\theta}],\\
\sigma_\theta^\alpha &= \lambda r^{\lambda-1}(A_\alpha e^{i(\lambda-1)\theta} + \bar a_\alpha \lambda e^{-i(\lambda-1)\theta} + \\
&\quad \bar b_\alpha e^{-i(\lambda+1)\theta}) + \bar\lambda r^{\bar\lambda-1}(a_\alpha e^{i(\bar\lambda-1)\theta} + \\
&\quad \bar A_\alpha \bar\lambda e^{-i(\bar\lambda-1)\theta} + \bar B_\alpha e^{-i(\bar\lambda+1)\theta}).
\end{aligned} \tag{7}$$

With $\lambda$ given in equation (4), the boundary condition of equation (2) may be satisfied with the arbitary choice of complex constants $A$ if we have

$$\begin{aligned}
&A_1 = A, a_1 = 0, B_1 = -A\lambda, b_1 = \bar A y_1,\\
&A_2 = Ay_2, a_2 = \bar A y_3, B_2 = Ay_4, b_2 = \bar A y_5,\\
&A_3 = Ay_6, a_3 = 0, B_3 = Ay_7, b_3 = \bar A y_8,
\end{aligned} \tag{8}$$

where $y_i (i = 1,\cdots,8)$ are the functions of the $\beta$ angle and the material elastic constants, given by

$$\begin{aligned}
y_1 &= -e^{-2i\lambda\pi},\\
y_2 &= [\mu_2(k_3 y_6 - \bar y_8 e^{-2i\lambda\beta}) + \\
&\quad \mu_3(y_6 - \bar y_8 e^{-2i\lambda\beta})]/[\mu_3(k_2 + 1)],\\
y_3 &= e^{-2i\lambda\beta}\bar y_7(\mu_3 - \mu_2)(e^{2i\beta} - 1)/[\mu_3(k_2 + 1)],\\
y_4 &= -y_6 e^{-2i\beta} + y_7 - y_2\lambda e^{-2i\beta} - \bar y_3 e^{-2i\lambda\beta},\\
y_5 &= \bar y_6 e^{-2i\bar\lambda\beta} + y_8 - \bar y_2 \lambda e^{-2i\lambda\beta} - y_3\lambda e^{-2i\beta},\\
y_6 &= [\mu_3(k_1 + e^{2i\lambda\pi}) + \mu_1(1 - e^{2i\lambda\pi})]/\\
&\quad [\mu_1(k_3 + 1)],\\
y_7 &= -y_6\lambda,\\
y_8 &= [\mu_1 k_3(1 - e^{-2i\bar\lambda\pi}) - \mu_3(k_1 + e^{-2i\bar\lambda\pi})]/\\
&\quad [\mu_1(k_3 + 1)].
\end{aligned} \tag{9}$$

From equations (7), (8) and (9), we have

$$\sigma_\theta^1 = \sigma_\theta^3 = A\lambda r^{(\lambda-1)}(1 - e^{-2i\lambda\pi}) \quad \text{at } \theta = 0. \tag{10}$$

We define the complex stress intensity factor $K = K_{\mathrm{I}} + iK_{\mathrm{II}}$ by

$$\sigma_\theta\Big|_{\theta=0} = Kr^{\lambda-1} \tag{11}$$

and observe that the complex constant $A$ in equation (8) and equation (10) can be expressed in terms of the stress intensity factor as

$$A = \frac{K}{\lambda(1 - e^{2i\lambda\pi})}. \tag{12}$$

## 2 Numerical Results and Discussion

**A.** If we consider the material 2 and material 3 same, with arbitary $\beta$ value, from equation (4) can have

$$\lambda = 1/2 + i\varepsilon \ , \quad \bar{\lambda} = 1/2 - i\varepsilon \ , \qquad (13)$$

where $\varepsilon = \ln R/(2\pi)$ ; $R = \dfrac{\mu_2 + \mu_1 k_2}{\mu_1 + \mu_2 k_1}$ ,

From equations (8) and (9), we have

$$A_1 = A, a_1 = 0, B_1 = -A\lambda, b_1 = \bar{A}/R,$$
$$A_2 = A_3 = A/R, a_2 = a_3 = 0, \qquad (14)$$
$$B_2 = B_3 = -A\lambda/R, b_2 = b_3 = \bar{A} \ ,$$

where

$$A = \frac{K}{\lambda(1+R)} \ .$$

This is just the case of the interface crack problem in bimaterial body[5] . Therefore the interface crack problem in bimaterial body can be consider as the special case of this paper.

**B.** As an application of previously theory, the $\lambda$ roots are calculated by numerical method in two cases: (1) $\mu_1/\mu_2 = 0.5$, $\nu_1 = \nu_2 = \nu_3 = 0.3$, $\beta = -135°$, $\mu_2/\mu_3$ varing from 1 to 10, the numerical results are sketched in Fig.2; (2) $\mu_1/\mu_2 = 0.5$, $E_2/E_3 = 4$, $\nu_1 = \nu_2 = \nu_3 = 0.3$, $\beta$ varing from 0° to $-180°$, the numerical results are sketched in Fig.3.

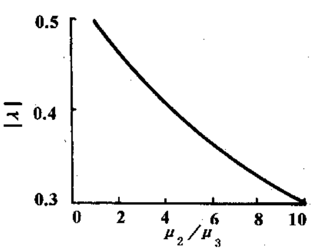


**Fig.2　Variation of $|\lambda|$ versus $\mu_2/\mu_3$ for $\beta = -150°$ and $\mu_1/\mu_2 = 0.5$**

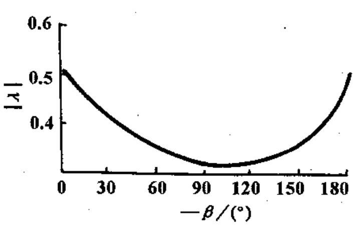


**Fig.3　Variation of $|\lambda|$ versus $-\beta$ for $\mu_1/\mu_2 = 0.5$ and $\mu_2/\mu_3 = 4$**

From Fig.2 and Fig.3, we may have the following conclusions:

(1) When the other factors are fixed, the $|\lambda|$ decreases with the increasing of $\mu_2/\mu_3$.

(2) When $\mu_2/\mu_3 = 1$, then $|\lambda| = 0.5$, this represents that the interface crack problem in bimaterial body is the special case of this paper.

(3) When the other conditions are fixed, the $|\lambda|$ varies largely with $\beta$ angle, which has the mininum value $\beta = -90°$.

(4) The $|\lambda|$ has same value with $\beta = 0$, or $\beta = -180°$, which shows that the $|\lambda|$ relates $\mu_i/\mu_j$ $(i, j = 1, 3)$, but the $|\mu_i|$ and $|\mu_j|$.

**C.** Now we consider a tri-material plate Fig.4 simillar to the examples[6] with a center crack which is 2a long. The composite plate is subjected to uniform tension $\sigma_0$. In case of $L = W = 1$, $a/W = 0.5$, $\nu_3 = 0.35$, $E_2/E_3$ varies from 1 to 10, we calculated the stress intensity factors $K_{\mathrm{I}}$, $K_{\mathrm{II}}$, $K_0$ and $K_0/K_\infty$, the numerical results are showed in Table 1 and sketched in Fig.5.

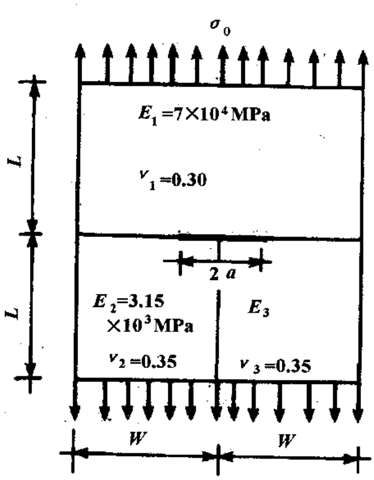


**Fig.4　Cracked composite plate**

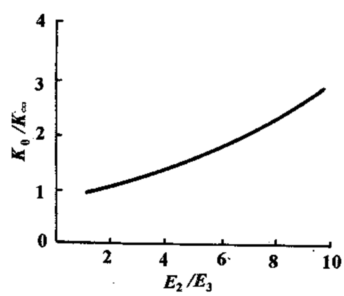


**Fig.5　Variation $K_0/K_\infty$ versus $E_2/E_3$ for cracked composite plate**

where

$$K_0 = |K| = \sqrt{K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2} \ , \qquad (15)$$

$$K_\infty = \sqrt{1 + 4\varepsilon^2}(\sigma_0/\sqrt{\alpha/2})^{[6]} . \qquad (16)$$

**TABLE 1 Stress intensity factors $K_{\mathrm{I}}$, $K_{\mathrm{II}}$, $K_0$ and $K_0/K_\infty$ for different $E_2/E_3$**

| $E_2/E_3$ | $K_{\mathrm{I}}$ | $K_{\mathrm{II}}$ | $K_0$ | $K_0/K_\infty$ | $K_0/K_\infty^{[6]}$ |
|---|---|---|---|---|---|
| 1 | 0.199475 | 0.039902 | 0.203423 | 0.901654 | 0.91 |
| 3 | 0.282581 | 0.045338 | 0.286195 | 1.268530 | — |
| 5 | 0.373602 | 0.082354 | 0.382569 | 1.695700 | — |
| 7 | 0.523107 | 0.133515 | 0.539834 | 2.392761 | — |
| 10 | 0.638440 | 0.181774 | 0.663813 | 2.942286 | — |

From Fig.5, the following conclusions may be drawn easily:

(1) When $E_1/E_2$ is the value 1, the plate consists of two material, the $K_0/K_\infty$ value 0.9017 approachs the value 0.91[6] much. This represents indirectly that the solution of this paper is correct.

(2) When $E_2/E_3$ increases, the $K_0/K_\infty$ increases also.

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# 三相异弹模薄板界面裂缝缝端应力奇异性与应力场


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**摘 要**:研究了三相异弹模薄板界面裂缝缝端奇异性与几何边界条件和弹性常数的依赖关系.采用原点设置在缝端的极坐标系统,用复势函数方法推导了缝端应力场的表达式,求得应力强度因子.算例表明,本文的解答是正确的,并得到一些有意义的结论.



**关键词**:应力奇异性;界面裂缝;复势函数方法