# 在有割缝区域上的哥西型积分的极限值

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已知，如果C是任一光滑闭曲线，函数$\varphi(z)$在C上解析，当C内部无割缝时，哥西型积分的极限值是：

$$\varphi_i(z_0)=\lim_{z\to z_0}F(z)=\frac{1}{2\pi i}\int_C\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta+\frac{1}{2}\varphi(z_0)$$

$$\varphi_e(z_0)=\lim_{z\to z_0}F_1(z)=\frac{1}{2\pi i}\int_C\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta-\frac{1}{2}\varphi(z_0)$$

其中F（z）定义在C的内部，$F_1(z)$定义在C的外部点$z_0$位于C上。

本文主要解决当C的内部区域具有割缝L时，哥西型积分的极限值公式仍成立。

以下分两种情况讨论。

**一、当点$z_0$位于割缝L的上、下岸时**

当点$z^0$位于割缝L的上岸时，作辅助积分路线：

$$APA'+A'B+BD+\widehat{DE}+EA$$

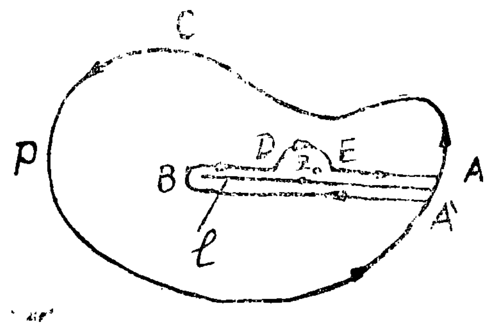


图　1

因函数

$$\frac{\varphi(\zeta)}{\zeta-z}$$

当点z在辅助积分路线外部时解析，由哥西积分定理

$$F_1(z)=\frac{1}{2\pi i}\int_{APA'}\frac{\varphi(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_{A'B}\frac{\varphi(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_{BD}\frac{\varphi(\zeta)}{\zeta-z}d\zeta$$

$$+\frac{1}{2\pi i}\int_{\widehat{DE}}\frac{\varphi(\zeta)}{t-2}d\zeta+\frac{1}{2\pi i}\int_{EA}\frac{\varphi(\zeta)}{\zeta-z}d\zeta$$

令$z\to z_0$，上式取极限得：

$$\varphi_e(z_0)=\lim_{z\to z_0}F_1(z)=\frac{1}{2\pi i}\int_{APA'}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta+\frac{1}{2\pi i}\int_{A'B}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta$$

$$+\frac{1}{2\pi i}\int_{BD}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta+\frac{1}{2\pi i}\int_{\widehat{BD}}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta+\frac{1}{2\pi i}\int_{EA}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta \qquad (1)$$

设$\widehat{DE}$是以$z_0$为心以$\varepsilon$为半径的圆弧，则有：

$$\zeta-z_0=\varepsilon e^{i\theta}\qquad(0\leqslant\theta\leqslant\pi)$$

$$d\zeta=i\varepsilon e^{i\theta}d\theta$$

因$\varphi(\zeta)$在$\widehat{DE}$上解析，故可设

$$\varphi(\zeta)=\varphi(z_0)+\eta$$

其中　　$|\eta|\to 0$，当$\varepsilon\to 0$时，从而可得

$$\frac{1}{2\pi i}\int_{\widehat{DE}}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta=\frac{1}{2\pi i}\int_{\pi}^{0}\frac{\varphi(z_0)+\eta}{\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta=\frac{\varphi(z_0)}{2\pi}\int_{\pi}^{0}d\theta$$

$$+\frac{1}{2\pi}\int_{\pi}^{0}\eta d\theta=-\frac{\varphi(z_0)}{2}+\frac{1}{2\pi}\int_{\pi}^{0}\eta d\theta$$

因

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\pi}^{0}\eta d\theta=0$$

所以

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\widehat{DE}}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta=-\frac{\varphi(z_0)}{2}$$

在（1）式左边令$\varepsilon\to 0$取极限，并注意这时D与E重合为一点，且积分在$A'B$与$BDA$上相反方向进行两次相互抵消，则得

$$\varphi_e(z_0)=\lim_{z\to z_0}F_1(z)=\frac{1}{2\pi i}\int_{APA'}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta-\frac{1}{2}\varphi(z_0)$$

$$=\frac{1}{2\pi i}\int_{C}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta-\frac{1}{2}\varphi(z_0)$$

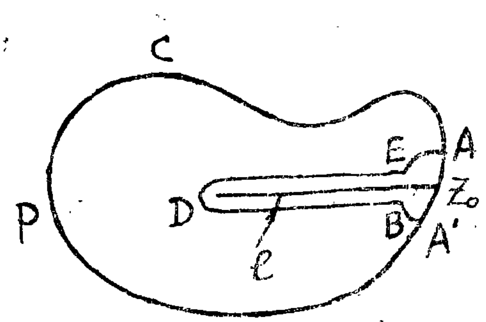


图　2

当点$z_0$位于割缝L的下岸时，只须把绕点$z_0$的园弧移到割缝L的下方，且积分沿顺时针方向从$2\pi$到$\pi$进行，亦得同样结果。

**二、当点$z_0$是割缝L与C的交点时**

**取辅助积分路线：**

$$APA'+\widehat{A'B}+BD+DE+\widehat{EA}$$

其中$\widehat{A'B}$与$\widehat{EA}$均为以$z_0$为心以$\varepsilon$为半径的四分之一园弧。

因函数

$$\frac{\varphi(\zeta)}{\zeta-z}$$

当点z位于辅助积分路线外部时解析，由哥西积分定理

$$F(z)=\frac{1}{2\pi i}\int_{APA'}\frac{\varphi(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_{\widehat{A'B}}\frac{\varphi(\zeta)}{\zeta-z}d\zeta$$

$$+\frac{1}{2\pi i}\int_{BD}\frac{\varphi(\zeta)}{\zeta-Z}d\zeta+\frac{1}{2\pi i}\int_{DE}\frac{\varphi(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_{\widehat{EA}}\frac{\varphi(\zeta)}{\zeta-z}d\zeta$$

令$z\to z_0$，上式取极限得：

$$\varphi_e(z_0)=\lim_{z\to z_0}F_1(z)=\frac{1}{2\pi i}\int_{APA'}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta+\frac{1}{2\pi i}\int_{\widehat{A'B}}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta+$$

$$\frac{1}{2\pi i}\int\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta+\frac{1}{2\pi i}\int_{DE}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta+\frac{1}{2\pi i}\int_{\widehat{EA}}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta\cdots\cdots \qquad (2)$$

由对$\widehat{A'B}$和$\widehat{EA}$的假设，有

$$\zeta-z_0=\varepsilon e^{i\theta}\qquad \zeta\in\widehat{A'B}\qquad \left(\pi\leqslant\theta\leqslant\frac{3\pi}{2}\right)$$

$$\zeta-z_0=\varepsilon e^{i\theta}\qquad \zeta\in\widehat{EA}\qquad \left(\frac{\pi}{2}\leqslant\theta\leqslant\pi\right)$$

$$d\zeta=i\varepsilon e^{i\theta}d\theta$$

因$\varphi(\zeta)$在$\widehat{A'B}$与$\widehat{EA}$上解析，故有$\eta_1$与$\eta_2$，使得

$$\varphi(\zeta)=\varphi(z_0)+\eta_1,\qquad \zeta\in\widehat{A'B},$$

$$\varphi(\zeta)=\varphi(z_0)+\eta_2,\qquad \zeta\in\widehat{EA}.$$

且当$\varepsilon\to 0$时，$|\eta_1|\to 0$，$|\eta_2|\to 0$，从而可得

$$\frac{1}{2\pi i}\int_{\widehat{A'B}}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta=\frac{1}{2\pi i}\int_{\frac{3\pi}{2}}^{\pi}\frac{\varphi(z)+\eta_1}{\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta=\frac{\varphi(z_0)}{2\pi}\int_{\frac{3\pi}{2}}^{\pi}d\theta+$$

$$\frac{1}{2\pi}\int_{\frac{3\pi}{2}}^{\pi}\eta_1 d\theta=-\frac{\varphi(z_0)}{4}+\frac{1}{2\pi}\int_{\frac{3\pi}{2}}^{\pi}\eta_1 d\theta$$

因 $$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\frac{3\pi}{2}}^{\pi}\eta_1 d\theta=0$$

所以

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\widehat{A'B}}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta=-\frac{\varphi(z_0)}{4}$$

同理可得 $$\lim_{\varepsilon\to 0}\int_{\widehat{EA}}\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta=-\frac{\varphi(z_0)}{4}$$

在（2）式左边令$\varepsilon\to 0$取极限，并注意这时B与A′重合，E与A重合，且积分在A′BD与DEA上相反方向进行两次相互抵消，则又得：

$$\varphi_e(z_0)=\lim_{z\to z_0}F_1(z)=\frac{1}{2\pi i}\int_{APA'}\frac{\varphi(\zeta)}{\zeta-z_0}-\left(\frac{\varphi(z_0)}{4}+\frac{\varphi(z_0)}{4}\right)$$

$$=\frac{1}{2\pi i}\int_C\frac{\varphi(\zeta)}{\zeta-z_0}d\zeta-\frac{\varphi(z_0)}{2}$$

如果点$z_0$不在L上，而位于C上，则可得到本文一开始指出的公式。

## 参 考 文 献

〔1〕 И.И.普里瓦洛夫著 《复变函数引论》 P191－195.

〔2〕 Н.И.穆斯海里什维里著 《奇异积分方程》 P46—53.