关于变形连续体虚功原理论证中平衡的必要与充分条件的表述

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本文对变形连续体静力平衡的必要与充分条件从运动微分方程的最一般的情况出发加以推证，以求得比较严格和直观的表述。

对处于动力载荷情形下的变形元体我们可写出其平衡方程如下：

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X&=\rho\ddot{u}\\ \frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+Y&=\rho\ddot{v}\\ \frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z&=\rho\ddot{w}\end{aligned}\right\}\quad(1)$$

式中ρ为该元体物质密度。$-\rho\ddot{u}$，$-\rho\ddot{v}$及$-\rho\ddot{w}$为元体惯性力分量。X，Y和Z代表体力分量。

对（1）式的三个平衡方程分别用虚位移δu，δv及δw相乘，并对整个体积V进行积分然后相加得

$$\int_V\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X\right)\delta u dV+\int_V\left(\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+Y\right)\delta v dV+\int_V\left(\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z\right)\delta w dV=\int_V\rho\left(\ddot{u}\delta u+\ddot{v}\delta V+\ddot{w}\delta w\right)dV\quad(2)$$

应用Green恒等式

$$\int_V\left(\frac{\partial\Phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial\psi}{\partial y}+\frac{\partial\Phi}{\partial z}\frac{\partial\psi}{\partial z}\right)dV=\int_S\Phi\left(l\frac{\partial\psi}{\partial x}+m\frac{\partial\psi}{\partial y}+n\frac{\partial\psi}{\partial z}\right)ds-\int_V\phi\nabla^2\psi dV$$

（2）式可改变成

$$\int_S\left[\left(\sigma_x l+\tau_{xy}m+\tau_{xz}n\right)\delta u+\left(\tau_{yx}l+\sigma_y m+\tau_{yz}n\right)\delta v+\left(\tau_{zx}l+\tau_{zy}m+\sigma_z n\right)\delta w\right]ds+\int_V\left(X\delta u+Y\delta v+Z\delta\omega\right)dV=\int_V\left[\sigma_x\delta\left(\frac{\partial u}{\partial x}\right)+\sigma_y\delta\left(\frac{\partial v}{\partial y}\right)+\right.$$

$$\sigma_z\delta\left(\frac{\partial w}{\partial z}\right)+\tau_{xy}\delta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{yz}\delta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial y}\right)+\tau_{zx}\delta\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\Big]dV$$

$$+\int_V\rho(\ddot{u}\delta u+\ddot{v}\delta u+\ddot{w}\delta w)dV \qquad (3)$$

在动力载荷的情况下变形体表面的应力平衡方程与静力载荷的情况相同，即

$$\left.\begin{aligned}\sigma_x l+\tau_{xy}m+\tau_{xz}n&=\bar{X}\\ \tau_{yx}l+\sigma_y m+\tau_{yz}n&=\bar{Y}\\ \tau_{zx}l+\tau_{zy}m+\sigma_z n&=Z\end{aligned}\right\} \qquad (4)$$

故（3）式左边可写成

$$\int_S\bar{X}^T\delta u ds+\int_V X^T\delta u dV。$$

其中

$$\bar{X}=\{\bar{X}\quad \bar{Y}\quad \bar{Z}\}$$

$$X=\{X\quad Y\quad Z\}$$

$$\delta u=\{\delta u\quad \delta v\quad \delta w\}$$

若尚有集中外荷载，在（3）式左边还应加上一项$P^T\delta U$.

其中

$$P=\{P_1\quad P_2\quad\cdots\cdots P_n\}$$

$$\delta U=\{\delta u_1\quad \delta u_2\quad\cdots\cdots\delta u_n\}$$ 为相应于P之位移变分。

（3）式右边两项可分别表示成

$$\int_V\sigma^T\delta\epsilon dV \quad 及 \int_V\rho\delta u^T\ddot{u}dV$$

其中

$$\sigma=\{\sigma_x\quad\sigma_y\quad\sigma_z\quad\tau_{xy}\quad\tau_{yz}\quad\tau_{zx}\}$$

$$\delta\epsilon=\left\{\delta\left(\frac{\partial u}{\partial x}\right)\quad\delta\left(\frac{\partial v}{\partial y}\right)\quad\delta\left(\frac{\partial w}{\partial z}\right)\quad\delta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\quad\delta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\quad\delta\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right\}$$

$$=\{\delta\varepsilon_{xx}\quad\delta\varepsilon_{yy}\quad\delta\varepsilon_{zz}\quad\delta r_{yx}\quad\delta r_{zy}\quad\delta r_{xz}\}$$

$\delta\varepsilon_{xx}$ $\delta\varepsilon_{yy}$ $\delta\varepsilon_{zz}$代表正应变变分，δr_{yx} δr_{zy} δr_{xz}代表剪应变变分。

$$\ddot{u}=\{\ddot{u}\quad\ddot{v}\quad\ddot{w}\}$$

于是（3）式可写成

$$\int_S\bar{X}^T\delta u ds+\int_V X^T\delta u dV-\int_V\rho\delta u^T\ddot{u}dV=\int_V\sigma^T\delta\epsilon dV \qquad (5)$$

下面根据（5）式推证必要与充分条件。

按变形体的虚功原理“变形体平衡的必要与充分条件是，对任意微小虚位移，外力所作总虚功等于变形体所接受的总虚变形功”。即

$$W_{ex}=U_{in} \tag{6}$$

式中$W_{ex}=\int_S \bar{X}^T\delta u ds+\int_V X^T\delta u dV$

$$U_{in}=\int_V \sigma^T\delta\epsilon dV$$

证明必要条件：若变形体处于静力平衡状态，即

$$\ddot{u}=0$$

由此得

$$\left.\begin{aligned}&\frac{\partial\sigma x}{\partial x}+\frac{\partial\tau xy}{\partial y}+\frac{\partial\tau xz}{\partial z}+X=0\\&\frac{\partial\tau xy}{\partial x}+\frac{\partial\sigma y}{\partial y}+\frac{\partial\tau yz}{\partial z}+Y=0\\&\frac{\partial\tau zx}{\partial x}+\frac{\partial\tau zy}{\partial y}+\frac{\partial\sigma z}{\partial z}+Z=0\end{aligned}\right\} \tag{7}$$

则（5）式成为

$$\int_S \bar{X}^T\delta u ds+\int_V X^T\delta u dV=\int_V \sigma^T\delta\epsilon dV \tag{8}$$

而等式左边等于W_{ex}，等式右边等于U_{in}，于是得证

$$W_{ex}=U_{in}$$

证明充分条件：若变形体之$W_{ex}=U_{in}$，则由（5）式

$$\int_V \rho\delta U^T\ddot{U}dV=0$$

由于$\rho\neq 0$以及δU的任意性，则必然$\ddot{U}\equiv 0$，然后由（8）式反过来应用Green恒等式可得

$$\int_V\left(\frac{\partial\sigma x}{\partial x}+\frac{\partial\tau xy}{\partial y}+\frac{\partial\tau xz}{\partial z}+X\right)\delta u dV+\int_V\left(\frac{\partial\tau yx}{\partial x}+\frac{\partial\sigma y}{\partial y}+\frac{\partial\tau yz}{\partial z}+Y\right)\delta v dV+$$

$$\int_V\left(\frac{\partial\tau zx}{\partial x}+\frac{\partial\tau zy}{\partial y}+\frac{\partial\sigma z}{\partial z}+Z\right)\delta w dV=0 \tag{9}$$

由于δu，δv及δw的任意性，可得出静力平衡条件（7）于是充分条件得证。

参 考 文 献

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